Convergence analysis and performance of an extended central force optimization algorithm

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ABSTRACT

Simple central force optimization (SCFO) algorithm is a novel physically-inspired optimization algorithm as simulating annealing (SA). To enhance the global search ability of SCFO and accelerate its convergence, a novel extended/enhanced central force optimization (ECFO) algorithm is proposed through both adding the historical information and defining an adaptive mass. SCFO and ECFO are all motivated by gravitational kinematics, in which the compound gravitation impels particles to the optima. The convergence of ECFO is proved based on a more complex characteristic equation than SCFO, i.e. the second order difference equation. The stability theory of discrete-time-linear system is used to analyze the motion equations of particles. Stability conditions limit their eigenvalues inside the unit cycle in complex plane and corresponding convergence conditions are deduced related with ECFO's parameters. Finally, ECFO are tested against a suite of benchmark functions with deterministic and excellent results. Experiments results show that ECFO converges faster than SCFO with higher global searching ability.

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1. Introduction

In the last decade of the 21st, various nature-inspired heuristic optimization algorithms became the most widely-used optimization methods [1]. Nature-inspired heuristic methods can be commonly divided into two kinds, biologically-inspired heuristics and physically-inspired heuristics, as they respectively imitate biological phenomena and physical principles [2,3]. Nowadays biologically-inspired heuristic optimization algorithms are applied widely in different areas [4–16], e.g. Genetic Algorithm (GA) [4], Differential Evolution (DE) [5], Artificial Immune System (AIS) [6], Ant Colony Optimization (ACO) [7], Bee Colony Algorithm (BCO) [8], Particle Swarm Optimization (PSO) [9–11], Shuffled Frog-leaping Algorithm [SFLA] [12], Fish Swarm Optimization (FSO) [9], Cat Swarm Optimization (CSO) [13], Bacterial Foraging Optimization (BFO) [14], Group Search Optimizer (GSO) [15] and Memetic Algorithms (MA) [16]. It is known that they all simulate biological interactive mechanisms, e.g. evolutionary, collective, competitive, collaborative, or swarm behaviors. Although certain behaviors are simulated quite perfectly, their deficiencies are still apparent. That is because the uncertainty of macro biological theories on micro individuals is evitable, such as randomness [4–16]. Consequently, the processes and results of optimization are all doubtful [17]; a high computational cost is needed; and a trivial statistical evaluation is indispensable. In addition, due to true random variables in underlying equations, they completely lack repeatability [18]. However, engineers and scientists have always been pursuing certain deterministic heuristic optimization algorithms with simple principles for

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various difficult problems. As a result, deterministic heuristics surge as an active research branch of optimization algorithms recently [17–32,35,36].

Ever since the famous simulating annealing (SA) that was proposed about 1980 [19], physically-inspired heuristic optimization algorithms have arouse great interests in the area of global optimization [17,18,20–32,35,36]. Nowadays many novel physically-inspired optimization algorithms have been proposed, e.g. Space Gravitational Optimization (SGO) [2], central force optimization (CFO) [17,20–28], Electromagnetism-like Mechanism (EM) [29], Artificial Physics Optimization (APO) [30], Gravitational Search Algorithm (GSA) [31] and Integrated Radiation Optimization (IRO) [32]. They are all based on deterministic physical principles. EM simulates the Coulomb's Force Law associated with electrical charge process. APO, GSA, SGO and CFO are based on Newton's law of gravity. Similarly, IRO introduced Einstein's general theory of relativity [33,34]. Although their performance may exceed traditional biologically-inspired heuristic optimization algorithms, the difficulty in traditional heuristic algorithms mentioned above is still persisting. Actually, besides SGO [2] and CFO [17,20–28,35,36], which inherently conduct deterministic computation, others are all stochastic like the traditional biologically-inspired random optimization algorithms [4–16]. SGO, CFO do not use any random operators to complete their optimizations [20–24,26]. Technically speaking, SGO is an embryonic form of CFO, as shown in [2].

Since CFO was proposed by R.A. Formato, it has been applied successfully in various real optimization problems, e.g. electromagnetic optimization [19], antenna array synthesis [26], training neural networks [27], detection, calibration in drinking networks [35] and parallel computing [36], which has shown high efficient and accurate as well as GA, PSO or GSO [17,36]. Now CFO is becoming a novel deterministic physically-inspired heuristic optimization algorithm [20–28,35,36]. Given a set of fixed parameters and an initial distribution, CFO will carry out deterministic search processes and the deterministic results are obtained for various optimization problems. Different from "evolve" operators in [4,5,37] or knowledge sharing in [5–16], particles in CFO shares global and individual information through gravitation, i.e. repulsion or attraction. And it converges rapidly to the optima with such a deterministic mechanical process. However, as CFO is proposed by R. A. Formato recently, most of published works pertain to him [17–25]. In addition, premature avoidance, convergence analysis, estimation of convergence rate, searching behaviors explanation, accelerating convergence and parameter selection are important problems to be analyzed but till now these works are scarce. In previous works, it can be seen that the original CFO in [16,19–27] is defined as simple or standard central force optimization (SCFO) algorithm, which is modeled in the first order ordinary difference dynamics as we proposed in [28].

Although SCFO converges rapidly as a result of the deterministic gravitational actions, it takes too much to compute the initial distributions [25,27,28,31]. Actually, exploitation of the initial distribution is realized by mass definition, which only depends on the larger in SCFO. Meanwhile, blindly ignoring others is unavoidable. The mass defined on USF (i.e. unit step function) on zero misses overall fitness information unexpectedly, which traps SCFO in local optima easily. Briefly, SCFO converges to the optima quickly at the beginning; but once trapped into local optima, it will stagnate and stop searching new space. So how to utilize the present initial distribution information throughout is crucial to avoid initializing the distribution in each process. For another thing, on computing the new acceleration in each process, SCFO discards the historical information of individuals and neglects the past global information. In essence, considering the characteristic equation, the flying orbits of particles in SCFO are short and instant responses of their initial positions [28]. It is detrimental to the efficiency of the global exploration of particles in SCFO.

No Free Lunch theorem [3] and Optimal Contraction Theorem [38] indicate that no optimizers can be optimal for arbitrary problems and a balance between exploitation and exploration for SCFO is what we desired for any problems. To enhance the searching ability of SCFO and accelerate its convergence, a novel CFO, named extended/enhanced central force optimization (ECFO) algorithm, is proposed here. Both local exploitation and global exploration are strengthened and balanced by two different methods, i.e. including each particle's historical acceleration and defining an adaptive mass. A new definition of mass is established on the basis of a new USF, which is regulated by an adaptive mean threshold. It expands the gravitational range of both larger and smaller particles with the manipulation in global exploitation. According to gravitational range analysis, the initial distribution of each process will be harnessed to its full potential. At the same time, a weighted historical experience is merged, which leads the order of characteristic equation of ECFO to be the second. So far, the searching behaviors of ECFO deserve a perfect explanation with a highly efficient global exploration. Clearly, the historical velocity information is the same as inertia item in PSO [10,11], i.e., the larger inertia weight achieves, the better global exploration is. Therefore, the exploitation of overall fitness information is expanded and futile searching attempts are avoided at a grand scale. Meanwhile, the cost of finding optimal initial distribution is reduced accordingly.

As we know, there is no other proof on the convergence of traditional deterministic CFO algorithm except the one in [28]. There are only two kinds of flying behaviors for particles to make a monotonous global searching with power decay or emanate. Using the stability theory of discrete-time-linear system to analyze the motion equations of particles in ECFO, the position responses of particles are the second order ordinary difference dynamics. There are ten kinds of particles flying behaviors at least. More indispensable convergence conditions are demonstrated with respect to the ECFO's parameters. This also provides the direction of analyzing and improving SCFO. What's more, a novel deterministic framework of CFO, named Dynamic Threshold Optimization (DTO) has been proposed [18]. An adaptive mean threshold is analyzed through gravitational range analysis in section 3.

The remainders of the paper are organized as follows. Brief concepts of SCFO are described in Section 2. Section 3 presents a novel extended/enhanced CFO, named ECFO. The comprehensive proof of ECFO is given in Section 4. Section 5 demonstrates the performance of ECFO and SCFO in a suite of benchmark functions. Section 6 presents conclusions and the future

research works with CFO. Not to confuse with each other, the concepts of "object", "mass", "individual", "particle" and "probe" are the same; while "generation", "step", "iteration" and "time" are the same.

2. Simple central force optimization (SCFO) algorithm

In SCFO, particles are attracted by gravitation based on the defined mass, which is similar to "virtual force" [30,31]. Particles are considered as objects and their performances are measured by the fitness function. In other words, each mass (object) represents a solution, which is navigated by adjusting the position properly according to the metaphorical principles.

Set the maximum object function $f(X)$: $\Omega \in R^{Nd} \to R$. $f(X)$ is the function to be maximized.

Define a bounded feasible region $\Omega = \{X|x_d^{min} < x_d < x_d^{max}, \ d = 1, \ldots, N_d\}$. Where N_d is the dimension of object function; x_d^{min}, x_d^{max} are the bound of dimension d. SCFO algorithm comprises three basic procedures [21–25,27,28]: (a) *Initialization*. (b) *Acceleration calculation*. (c) *Motion*. A general framework of SCFO is shown in Table 1.

In *initialization* procedure, a population of particles is created in N_d-dimension space. An initial distribution is formed by deploying N_p/N_d particles uniformly on each "probe lines" determined by distribution factor γ [17], where N_p denotes the number of particles. The initial acceleration is set to zero. Under a predefined distribution, SCFO searches the optima with a deterministic method as described in (b), (c).

The next is *acceleration calculation*. The compound acceleration of one particle from components in each direction is calculated according to the metaphorical principle of Newton's law of gravity [27,30,31]. Mass is a user-defined function from the object function to be maximized. To avoid missing the landscapes of problems, the simplest definition of mass in [30] is used. In gravitational field of particle p, the mass of particle k at j can be expressed:

$$U\left(M_{j-1}^k - M_{j-1}^p\right)\left(M_{j-1}^k - M_{j-1}^p\right)^\alpha, \quad k = 1, \ldots, p-1, p+1, \ldots, N_p \tag{1}$$

SCFO's mass USF:

$$U(z) = \begin{cases} 1, & z \geqslant 0 \\ 0, & else \end{cases} \tag{2}$$

Where $U()$ is an USF on zero, $M_{j-1}^p = f\left(x_1^{p,j-1}, x_2^{p,j-1}, \ldots, x_{N_d}^{p,j-1}\right)$ is the object function value of particle p at $j-1, j \in \{1, \ldots, Nt\}$, N_t are total iterations. For other mass definitions, please refer to [30] with different performances. Generally, acceleration of particle p at step $j-1$ is presented.

Table 1
An algorithmic description of the SCFO algorithm.

Step 1: Initialization A
Set decision space Ω's dimension N_d, boundaries X_{min}, X_{max}; Set total number of particles N_p, total number of time steps N_t, CFO's gravitational constant G, mass related parameters α, β, particle distribution factor γ and repositioning factor ΔF_{rep};
Step 2: Initialization B
Compute initial probe distribution X with increment γ, initial fitness matrix M, and initial G; Assign initial particle acceleration A; Set initial F_{rep}
Step 3: Evaluate earlier termination criterion or less than N_t
If reach N_t, return to **Step 2**; If fulfill stop criterion, return **Step 4**
Step 3.1: Compute probe position vectors
Step 3.2: Retrieve errant particles
Step 3.3: Compute fitness matrix of current particle distribution and finesses
Step 3.4: Increment F_{rep}, update G
Step 3.5: Shrink Ω^1
Return **Step 3**
Step 4: Stop and putout best solution have reached so far

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[1] Shrinks the decision space to the neighborhood of best particle [17]. To avoid premature and stagnation, we abandons it here.

$$\overrightarrow{A}_{j-1}^p = G\sum_{\substack{k=1\\k\neq p}}^{N_p}\overrightarrow{A}_{j-1}^{k,p} = G\sum_{\substack{k=1\\k\neq p}}^{N_p}\left[U\left(M_{j-1}^k - M_{j-1}^p\right)\left(M_{j-1}^k - M_{j-1}^p\right)^\alpha \times \frac{\left(\overrightarrow{X}_{j-1}^k - \overrightarrow{X}_{j-1}^p\right)}{\left\|\overrightarrow{X}_{j-1}^k - \overrightarrow{X}_{j-1}^p\right\|^\beta}\right] \tag{3}$$

Where $\overrightarrow{A}_{j-1}^{k,p}$ is an accelerate vector of particle k toward to particle p, $k = 1$ and $k \neq p$. In Eq. (3), G, α and β do not represent the concrete gravitational fundamentals.

The final procedure is *Motion*. According to the acceleration calculated previously, the positions and "velocities" of particles are updated based on Newton's motion laws [27,30,31]. If acceleration $\overrightarrow{A}_{j-1}^p$ is exerted, particle p will move from $\overrightarrow{X}_{j-1}^p$ to $\overrightarrow{X}_j^p$ according to the motion equation.

$$\overrightarrow{X}_j^p = \overrightarrow{X}_{j-1}^p + 0.5 \times \overrightarrow{A}_{j-1}^p \Delta t^2, \quad j \geqslant 1 \tag{4}$$

Where Δt represents the interval, and 0.5 illustrates the kinematic metaphor. The positions are updated based on last "mass" information as a deterministic gradient algorithm. Movement of each particle is restricted to the bounded feasible region. However, when some "fly" out, a necessary retrieving mechanism in [17] is used to act on them, i.e. replacing errant particles according to the last positions.

After updating the positions of all particles by sharing the gravitational information, the object function is updated at the new positions. Because under a predefined initial distribution SCFO can definitely reach a stable state after computing the best, it will update a new initial distribution to restart until the global best or a desirable solution is reached. The convergence conditions of SCFO in [28] have revealed that it will converge to the optima that have searched so far, which is not worse than the predefined one in initial distributions.

3. The Extended/Enhanced CFO (ECFO) algorithm

ECFO utilizes a new landscape of mass by an USF based on an adaptive mean threshold. It expands the gravitational ranges of particles according to the following gravitational range analysis; the maximum utilization of initial distributions can be made; meanwhile it avoids futile attempts for a local exploitation. On the other hand, ECFO includes a weighted historical experience by adding historical acceleration. A complex dynamic mechanical characteristic of particles flying in decision space is revealed in Remark 3.4 and proved in Section 4.

Three kinds of masses defined in theoretical physics are active gravitational mass, passive gravitational mass and inertial mass [33,34]. If only consider larger objects, the mass in SCFO is positive, which is similar to what the weak equivalence principle shows in [34]. Conceptually all objects can attract or repel one another, and the force between them is interactive. A particle with positive mass will possess a positive gravitational field to attract others. Comparatively a particle with passive mass will possess a passive gravitational field to repel others. The inertia mass is against the motion and makes objects moving slowly.

In SCFO, the mass is based on relative fitness value [21–25] instead of the passive gravitational mass. To make a distinct comparison, a brief format refers to Eqs. (1) and (2). Only when the fitness is larger than particle p, will the attractive gravitation be exerted on p. SCFO converges to local optima quickly so as to lose fitness information overall. An important mechanism in SCFO is that the best cannot be attracted by others. What is more, in order to reach the overall optima, a larger computation for updating initial distribution is always indispensable.

Unlike the fixed positive mass in SCFO, ECFO utilizes a new landscape of mass by defining an USF, which is based on an adaptive mean threshold. The total relative masses will adjust it to different particles distributions adaptively. An adaptive USF in the definition of mass for particle p can be presented.

The adaptive mean threshold:

$$St = \frac{1}{N_p - 1}\sum_{\substack{k=1\\k\neq p}}^{N_p}\left(M_{j-1}^k - M_{j-1}^p\right) \tag{5}$$

ECFO's new mass USF:

$$U(z) = \begin{cases} 1, & z \geqslant -St \\ 0, & else \end{cases} \tag{6}$$

Therefore, it includes three gravitational mass with three gravitational actions, attraction, repulsion and persistence. It expands the gravitational range of both larger and smaller particles as analyzed in followings with the exploitation of overall mass information. To explain the characteristics of ECFO compared with SCFO, two typical cases are shown in Figs. 1 and 2, where a larger back cycle corresponds to greater fitness, shadow cycles are implicit. Gravitational range is an important metric of the exploitation for larger particles or smaller particles. Consider the range of gravitational field with both the new defined mass in ECFO and the traditional one. The limits of the gravitational ranges are the best one and the worst one. Here, the best or worst denote one particle owns the maximum or minimum fitness with a certain particles distribution. Now, four definitions are given.
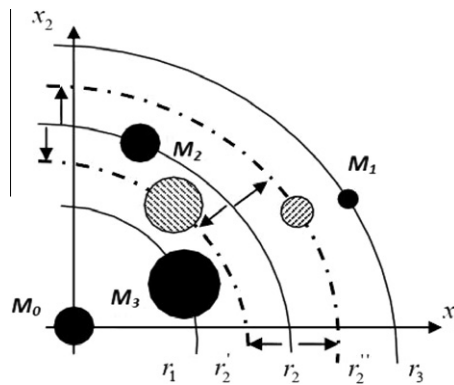
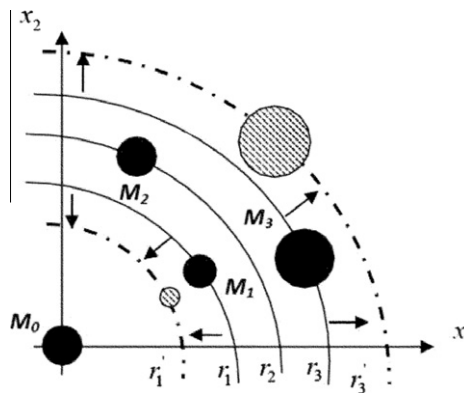**Fig. 1.** Expanding or shrinking gravitational range of M_0.

Fig. 2. Expanding or shrinking gravitational range of M_0.

Definition 3.1 (*Maximum Gravitational Range (MaxGR)*). The maximum gravitational range of a particle is the maximum distance from the others, in which the gravitational attraction or repulsion is valid to the particle. □

Definition 3.2 (*Minimum Gravitational Range (MinGR)*). The minimum gravitational range of a particle is the minimum distance from the others, in which the gravitational attraction or repulsion is valid to the particle. □

Definition 3.3 (*Equivalent Gravitational Range (EqGR)*). The equivalent gravitational range of a particle is the distance from the others, in which the fitness is equivalent to its. If such others exist, or not only one, select the maximum range as *EqGR*; if not exist, *EqGR* is arithmetic mean of *MinGR* and *MaxGR* (exists a visual equivalent one). □

Definition 3.4 (*Gravitational Range (GR)*). The gravitational region of a particle is the distance between *MaxGR*'s or*MinGR*'s circle and*EqGR*'s circle or both. □

Theorem 3.1. *In SCFO, under a certain particles distribution, for the best, exists GR = MinGR = MaxGR = EqGR = ∅; for the worst, exists GR = Ω.*

Proof. According to Eqs. (1) and (2) in SCFO, the best keeps still, not be attracted or repelled by others. So, its gravitational range is empty. Another extreme is that the worst will be affected by others. Maximum gravitational range of the worst is the distance to the furthest, namely overall searching space. Minimum gravitational range of the worst is the distance to the nearest particle. Thus, the worst probably walk though the total feasible region. □

Theorem 3.2. *In ECFO, under a certain particles distribution, for the best, its EqGR equals to the mean of MinGR and MaxGR, which exists non-empty; for the worst, exists GR = Ω.*

Proof. According to Eqs. (3), (5) and (6) in ECFO, the best will be attracted or repelled by others. Both *MinGR* and *MaxGR* exist non-empty. So, the gravitational range of the best is nonempty. Another extreme is the same as SCFO. Maximum gravitational range of the worst equals to one in SCFO. Minimum gravitational range of the worst may shrink to the nearest in SCFO. Thus, the worst may walk though the region as SCFO at least. □

Remark 3.1 (*The best and the worst in both SCFO and ECFO*). For the best in SCFO, it persists until a better one is found as the elitist preservation in GA [4]. Although the preservation can accelerate its convergence, it always results in premature for multimodal functions. To relax the best, an adaptive mass in ECFO enables it to tune their positions locally for the better. For the worst, although the gravitational range of both SCFO and ECFO is Ω, maximum gravitational range in ECFO varies adaptively to the overall fitness information as Eq. (5) not the fixed in SCFO. As a result, the local exploitation and global exploration of SCFO are augmented in ECFO. □

Two typical cases are about the mediate, which principally determine the convergence speed and the quantity of the optimal. The maximum utilization of initial distributions is largely up to them. Proper adjustment of mediate gravitational ranges is crucial for balancing exploration and exploitation. Figs. 1 and 2 show two extreme 2-D CFO systems, where M_0, M_1, M_2 and M_3 represent four particles with $M_1 < M_0 = M_2 < M_3$. Without loss of generality, other particles are simplified as M_3, which denotes the particles whose fitness is larger than M_0; and M_1 denotes particles with fitness smaller than M_0. Considering how others affect the motion of M_0, *GR*, *MaxGR*, *MinGR*, *EqGR* are presented. To compare with SCFO, an equivalent gravitational range of M_0 is drawn as $EqGR = r_2$ in Figs. 1 and 2.

Remark 3.2 (*A Comparison in Fig. 1*). According to Eqs. (1) and (2) in SCFO, as only $M_3 > M_0$, exists

$$MinGR = MaxGR = r_1, EqGR = r_2, \tag{7}$$
$$GR = EqGR - MaxGR = r_2 - r_1. \tag{8}$$

Accordingly M_0 can be trapped into the position of M_3 easily based on the local mass landscape in SCFO in despite of a nearest and rapid movement. Comparatively, ECFO can access the overall fitness information to get a mean mass landscape. According to Eqs. (5) and (6), there are two situations: (i) and (ii).

(i) As $|M_3 - M_0| > |M_1 - M_0|$, thus $St > 0$. $MaxGR(M_0)$ expands from r_2 in SCFO to r_2'' in ECFO with a wider mass landscape, exists

$$MinGR = r_1, MaxGR = r_2'', EqGR = r_2. \tag{9}$$
$$GR = MaxGR - MinGR = r_2'' - r_1. \tag{10}$$

(ii) As $|M_1 - M_0| > |M_3 - M_0|$, thus $St < 0$. $MinGR(M_0)$ shrinks from r_2 in SCFO to r_2' in ECFO to avoid a relative smaller local minimum M_1, exists

$$MinGR = r_1, MaxGR = r_2', EqGR = r_2. \tag{11}$$
$$GR = MaxGR - MinGR = r_2' - r_1. \tag{12}$$

Certainly if they are equivalent, $GR(M_0)$ maintains as one in SCFO. In conclusion, M_0 expands its gravitational range to avoid being trapped in local optima as M_3 in Eq. (10) and shrinks it to prevent falling into the worst as M_1 in Eq. (12). □

Remark 3.3 (*A Comparison in Fig. 2*). A similar analysis of Fig. 2 is described. According to Eqs. (1) and (2), exists

$$MaxGR = r_3, MinGR = EqGR = r_2. \tag{13}$$
$$GR = MaxGR - EqGR = r_3 - r_2. \tag{14}$$

According to Eqs. (5) and (6), exists two situations: (i) and (ii).

(i) As $|M_3 - M_0| > |M_1 - M_0|$, thus $St > 0$. M_0 may search carefully in range r_1' or r_1 to r_3 not as r_2 to r_3 in SCFO, because M_1 may slow it to jump out of range r_2, exists

$$MaxGR = r_3, MinGR = r_1', EqGR = r_2. \tag{15}$$
$$GR = MaxGR - MinGR = r_3 - r_1'. \tag{16}$$

(ii) As $|M_1 - M_0| > |M_3 - M_0|$, thus $St < 0$. $MaxGR(M_0)$ expands from r_3 in SCFO to r_3' in ECFO to avoid a relative smaller local minimum M_1, exists

$$MaxGR = r_3', MinGR = EqGR = r_2. \tag{17}$$
$$GR = MaxGR - MinGR = r_3' - r_2. \tag{18}$$

Therefore, M_0 expands its gravitational range to avoid falling into the worst as M_1 and get a more careful local search to avoid premature. □

From Remarks 3.2 and 3.3, USF in ECFO as Eqs. (5) and (6) improves the mass landscape in SCFO. As a result, it expands or shrinks the gravitational range adaptively to avoid premature or the worst and enhances both the local and global search ability. As in each process the particles distribution can be treated as initial distribution, above analysis works for any time. So the initial distribution is harnessed to its potential. A dynamic balance between local exploitation and global exploration is achieved. So the "Extend" or "Enhanced" characteristic of ECFO is apparent compared to SCFO.

To be further, another big difference between SCFO and ECFO is the order of the characteristic equation of particles motion. It can be seen easily that whether the historical velocity item exists in Eq. (4) or not. A classic or basic description of SCFO you can find in [21–25] and its convergence analysis you can find in [28], pseudo random CFO (PR-CFO) [22] and parameter-free CFO (PF-CFO) [24] are also based on such a framework. Although Newton's motion law shows velocity item is necessary, SCFO does not adopt it for simplicity [17,21–28]. The historical velocity information, i.e. last initial "velocity", is the same as inertia item in PSO. The larger inertia weight achieves the better global exploration [10,11], which changes the dynamic searching process intrinsically. ECFO includes a weighted historical experience by adding historical acceleration.

$$\overrightarrow{X}_j^p = \overrightarrow{X}_{j-1}^p + \overrightarrow{A}_{j-2}^p \Delta t + 0.5 \times \overrightarrow{A}_{j-1}^p \Delta t^2 \qquad (19)$$

Remark 3.4 (*Complex Dynamics*). Following Theorem 4.1 demonstrates that the flying orbits of particles in ECFO are more complex than SCFO, i.e. the second order difference dynamic processes. As we all known, the main dynamic characteristics of a second order difference system are emanate amplitude oscillate, equivalent amplitude oscillate, attenuate amplitude oscillate, power decay and power amplify. So particles in ECFO own much more searching behaviors than SCFO. To be exact, two opposite mechanical actions, attraction and repulsion will duplicate such searching behaviors. What's more, although particles motion in ECFO is a linear-time-invariant difference equation at each step, the property of varying parameters of characteristic equation will complicate the simple deterministic searching processes. □

4. Convergence analysis

The convergence proof of ECFO is developed here. It is crucial for understanding the complex dynamic characteristics and making a further improvement. It reveals the necessary convergence conditions under which ECFO is guaranteed to converge to the optima. It brings to light that its global convergence mainly depends on the utilization of initial distributions. This proof also encourages further work of analyzing and improving the framework of CFO [18,21–25,27,28], especially DTO in [18].

Because ECFO is inherently deterministic under a predefined initial distribution, the deterministic sequence convergence definition is used directly.

Definition 4.1 (Deterministic Sequence Convergence [30]). A deterministic discrete-time sequence $\{X(j)\}$, $j = 0, 1, \ldots$ of scalars or vectors $X(j)$ converges to the constant value X^* if the limit exists: $\lim_{t\to\infty} X(j) = X^*$. The limit X^* may not be a prior-known solution. □

Following symbols are in accordance with definitions before. Without loss of generality, particle i is selected arbitrarily and a 1-D ECFO is analyzed, $N_d = 1$. Assume the feasible region is always positive.

Theorem 4.1. *ECFO's motion is a secondary difference dynamic process in essence.*

Proof. According to Eq. (6), define $M_i = \{k|f(X_k) - f(X_i) > -St, k = 1, \ldots, N_p\}$. Eq. (3) is rewritten to

$$A_i(j) = G \cdot \sum_{k \in M_i} \frac{[f(X_k) - f(X_i)]^\alpha}{\|X_k - X_i\|^\beta}(X_k(j) - X_i(j)) = G \cdot \sum_{k \in M_i} \frac{[f(X_k) - f(X_i)]^\alpha}{\|X_k - X_i\|^\beta}X_k(j) - G \cdot \sum_{k \in M_i} \frac{[f(X_k) - f(X_i)]^\alpha}{\|X_k - X_i\|^\beta}X_i(j). \qquad (20)$$

With the following definitions

$$\phi_i(j) = G \cdot \sum_{k \in M_i} \frac{[f(X_k) - f(X_i)]^\alpha}{\|X_k - X_i\|^\beta}X_k(j); \qquad (21)$$

$$\theta_i = G \cdot \sum_{k \in M_i} \frac{[f(X_k) - f(X_i)]^\alpha}{\|X_k - X_i\|^\beta}. \qquad (22)$$

Eq. (3) is updated to

$$A_i(j) = \phi_i(j) - \theta_i \cdot X_i(j). \qquad (23)$$

Together with Eq. (19), SCFO system equations are obtained. ($\Delta t \neq 0$ is obvious.)

$$A_i(j) = \phi_i(j) - \theta_i \cdot X_i(j); \tag{24}$$

$$X_i(j+1) = X_i(j) + \frac{1}{2}A_i(j) \cdot \Delta t^2 + A_i(j-1) \cdot \Delta t. \tag{25}$$

Assume that the fitness landscape is so smooth that θ_i, ϕ_i keep stable in two intervals. Substitute Eq. (24) into Eq. (25) to eliminate $A_i(j)$ and $A_i(j-1)$, one has

$$X_i(j+1) + \left(\frac{1}{2} \cdot \Delta t^2 \theta_i - 1 \right) X_i(j) + \theta_i \Delta t \cdot X_i(j-1) = \frac{1}{2} \phi_i(j) \Delta t^2 + \phi_i(j-1) \Delta t. \tag{26}$$

Because ϕ_i and θ_i are constants as assumed against particle i at each generation, Eq. (26) is a second order ordinary difference equation between step j and $j-1$.   □

To access to following analysis fluently, eigenvalues conditions are partially cited as Theorem 4.2.

Theorem 4.2 (Eigenvalues Conditions [39]). *The discrete-time linear-time-invariant system is (i) marginally stable iif all eigenvalues of system have magnitude smaller than or equal to 1; (ii) asymptotically and exponentially stable iif all eigenvalues of system have magnitude strictly smaller 1; (iii) unstable iif at least one eigenvalues of system has magnitude larger than 1 or magnitude equal to 1.*   □

According to Theorem 4.2, the stability condition of discrete-time-linear system is that its eigenvalues lie inside the unit cycle on the complex plane. Obviously solutions of Eq. (26) converge to a deterministic limit. The necessary convergence conditions are proved in Theorem 4.3. To simplify Theorem 4.3, three lemmas are taken into forwards.

The characteristic equation of Eq. (26) is

$$\lambda^2 + \left(\frac{1}{2} \cdot \Delta t^2 \theta_i - 1 \right) \lambda + \theta_i \Delta t = 0. \tag{27}$$

According to the eigenvalues condition, i.e. the absolute value of eigenvalues λ_1, λ_2 is less than 1.

$$|\lambda_{1,2}| = \frac{1}{2} \cdot \left| \left(1 - \frac{1}{2} \cdot \Delta t^2 \theta_i \right) \pm \sqrt{ \left(\frac{1}{2} \cdot \Delta t^2 \theta_i - 1 \right)^2 - 4 \cdot \theta_i \Delta t } \right| < 1. \tag{28}$$

Obviously $\Delta t \neq 0$, and set $\theta_i \neq 0$. There three cases are analyzed in Lemmas 4.1, 4.2, and 4.3.

Lemma 4.1. *Given $\Delta t \neq 0$ and $\theta_i \neq 0$, for Eq. (27), exists two equivalent eigenvalues inside unit cycle in complex plane, which means Eq. (26) is stable, iif they satisfy one condition of followings*

(i) $\theta_i = 2 \cdot \frac{(\Delta t + 4) - \sqrt{8\Delta t + 16}}{\Delta t^3}$, $\Delta t > 0 \vee -2 < \Delta t < 0$;
(ii) $\theta_i = 2 \cdot \frac{(\Delta t + 4) + \sqrt{8\Delta t + 16}}{\Delta t^3}$, $\Delta t > 6$.

Proof. Actually, it requires Eq. (27) has two equivalent eigenvalues. Thus $\left(\frac{1}{2} \cdot \Delta t^2 \theta_i - 1 \right)^2 = 4 \cdot \theta_i \Delta t$, that is

$$\frac{1}{4} \cdot \Delta t^4 \theta_i^2 - (\Delta t^2 + 4\Delta t) \theta_i + 1 = 0 \tag{29}$$

Regard θ_i as unknown in Eq. (29). If $\Delta t \geqslant -2$, exists two roots $s_{1,2} = 2 \cdot \frac{(\Delta t + 4) \pm \sqrt{8\Delta t + 16}}{\Delta t^3}$.
According to the stability condition, for Eq. (27), one has

$$|\lambda_1| = |\lambda_2| = \frac{1}{2} \left| 1 - \frac{1}{2} \cdot \Delta t^2 \theta_i \right| < 1. \tag{30}$$

Then to simplify Eq. (30) with Eq. (29), thus $\sqrt{4\Delta t \theta_i} < 2$. So, there are two possible cases with Eq. (30), (a) and (b).

(a) If $-2 \leqslant \Delta t < 0$, and $\theta_i < 0$, only exists $\frac{1}{\Delta t} < s_2 < 0$.
(b) If $\Delta t > 0$, and $\theta_i > 0$, exists $0 < s_2 < \frac{1}{\Delta t}$. Only if $\Delta t > 6$, exists $0 < s_1 < \frac{1}{\Delta t}$. Combine (a) and (b), lemma 1 is obtained.   □

Lemma 4.2. *Given $\Delta t \neq 0$ and $\theta_i \neq 0$, for Eq. (27), exists two complex eigenvalues inside unit cycle in complex plane, which means Eq. (26) is stable, iif they satisfy one condition of followings*

(i) $\frac{1}{\Delta t} < \theta_i < 2 \cdot \frac{(\Delta t + 4) - \sqrt{8\Delta t + 16}}{\Delta t^3}$, $-2 \leqslant \Delta t < 0$;
(ii) $2 \cdot \frac{(\Delta t + 4) - \sqrt{8\Delta t + 16}}{\Delta t^3} < \theta_i < \frac{1}{\Delta t}$, $0 < \Delta t < 6$;
(iii) $2 \cdot \frac{(\Delta t + 4) - \sqrt{8\Delta t + 16}}{\Delta t^3} < \theta_i < 2 \cdot \frac{(\Delta t + 4) + \sqrt{8\Delta t + 16}}{\Delta t^3}$, $\Delta t \geqslant 6$.

Proof. Actually, it requires Eq. (27) has two complex eigenvalues with the same magnitude. Consider Eq. (28), as $\left(\frac{1}{2} \cdot \Delta t^2 \theta_i - 1\right)^2 < 4 \cdot \theta_i \Delta t$, exists

$$\frac{1}{4} \cdot \Delta t^4 \theta_i^2 - (\Delta t^2 + 4\Delta t)\theta_i + 1 < 0. \tag{31}$$

If $\Delta < 0$ of Eqs. (29), (31) is impossible. If $\Delta \geqslant 0$, that is $\Delta t \geqslant -2$, $\Delta t \neq 0$. Consider two roots of Eq. (29), two situations (a) and (b) are given with Eq. (31).

- (a) If $-2 \leqslant \Delta t < 0$, exists $s_1 < \theta_i < s_2$;
- (b) If $\Delta t > 0$, exists $s_2 < \theta_i < s_1$.
  Actually, stability condition requires $|\lambda_1| = |\lambda_2| < 1 \Longleftrightarrow \sqrt{4\theta_i \Delta t} < 2$. Thus (a) and (b) corresponds situations (c) and (d) respectively.
- (c) If $-2 \leqslant \Delta t < 0$, exists $\frac{1}{\Delta t} < \theta_i < 0$. With $\frac{1}{\Delta t} > s_1$, combine (a), the stability condition requires $\frac{1}{\Delta t} < \theta_i < s_2$. The condition (i) is obtained;
- (d) If $0 < \Delta t$, exists $0 < \theta_i < \frac{1}{\Delta t}$, two intervals (d1) and (d2) are considered.
- (d1) If $0 < \Delta t < 6$, with $\frac{1}{\Delta t} < s_1$, combine (b), the stability condition requires $s_2 < \theta_i < \frac{1}{\Delta t}$. The condition (ii) is obtained;
- (d2) If $\Delta t \geqslant 6$, with $\frac{1}{\Delta t} \geqslant s_1$, combine (a), the stability condition requires $s_2 < \theta_i < s_1$. The condition (iii) is obtained. So far three conditions are obtained completely. □

Lemma 4.3. *Given $\Delta t \neq 0$ and $\theta_i \neq 0$, for Eq.* (27)*, exists two different real eigenvalues inside unit cycle in complex plane, which means Eq.* (26) *is stable, iif they satisfy one condition of followings*

- (i) $0 < \theta_i < \frac{2}{\Delta t^2}, \frac{2}{\Delta t^2} < \theta_i < \frac{4}{\Delta t^2 - 2\Delta t}, \ \Delta t < -2$;
- (ii) $\theta_i < 2 \cdot \frac{(\Delta t + 4) + \sqrt{8\Delta t + 16}}{\Delta t^3}, 2 \cdot \frac{(\Delta t + 4) - \sqrt{8\Delta t + 16}}{\Delta t^3} < \theta_i < 0, \ -2 \leqslant \Delta t < 0$;
- (iii) $0 < \theta_i < 2 \cdot \frac{(\Delta t + 4) - \sqrt{8\Delta t + 16}}{\Delta t^3}, \ \Delta t > 0$;
- (iv) $2 \cdot \frac{(\Delta t + 4) + \sqrt{8\Delta t + 16}}{\Delta t^3} < \theta_i, \ 0 < \Delta t < 2$;
- (v) $2 \cdot \frac{(\Delta t + 4) + \sqrt{8\Delta t + 16}}{\Delta t^3} < \theta_i < \frac{4}{\Delta t^2 - 2\Delta t}, \ \Delta t > 2$.

Proof. Actually, it requires Eq. (27) has two different real eigenvalues. Consider Eq. (28), as $\left(\frac{1}{2} \cdot \Delta t^2 \theta_i - 1\right)^2 > 4 \cdot \theta_i \Delta t$, exists

$$\frac{1}{4} \cdot \Delta t^4 \theta_i^2 - (\Delta t^2 + 4\Delta t)\theta_i + 1 > 0. \tag{32}$$

For Eq. (29), it is assigned into two aspects, $\Delta < 0$ and $\Delta \geqslant 0$ i.e. (a) and (b).

- (a) If $\Delta < 0$, that is $\Delta t < -2$, θ_i in Eq. (32) exists anyway. By inspecting Eq. (29), exists four situations (a1), (a2), (a3) and (a4).
- (a1) If $\theta_i > 0$, and $1 - \frac{1}{2} \cdot \Delta t^2 \theta_i > 0$, exists $\lambda_2 < 0 < \lambda_1, |\lambda_1| > |\lambda_2|$, so the stability condition requires $\lambda_1 < 1$.
- (a2) If $\theta_i > 0$, and $1 - \frac{1}{2} \cdot \Delta t^2 \theta_i < 0$, exists $\lambda_2 < 0 < \lambda_1, |\lambda_1| < |\lambda_2|$, so the stability condition requires $\lambda_2 > -1$.
- (a3) If $\theta_i < 0$, and $1 - \frac{1}{2} \cdot \Delta t^2 \theta_i > 0$, exists $\lambda_1 > \lambda_2 > 0$, so the stability condition requires $\lambda_1 < 1$.
- (a4) If $\theta_i < 0$, and $1 - \frac{1}{2} \cdot \Delta t^2 \theta_i < 0$, exists $\lambda_2 < \lambda_1 < 0$, so the stability condition requires $\lambda_2 > -1$.

Consider (a1) and (a3), exists $0 < \lambda_1 < 1$. Note $\Delta t < -2$, exists $\theta_i > 0$, so (a3) is excluded. In (a1), note $1 - \frac{1}{2} \cdot \Delta t^2 \theta_i > 0$, the stability condition requires

$$0 < \theta_i < \frac{2}{\Delta t^2}, \quad \Delta t < -2. \tag{33}$$

Next for (a2), exists $-1 < \lambda_2 < 0$. Obviously, exists $\theta_i < \frac{6}{\Delta t^2}$, otherwise it is false. Note $\Delta t < -2$, exists $\theta_i < \frac{4}{\Delta t^2 - 2\Delta t} (> 0)$. So, for (a2), exists $0 < \theta_i < \frac{4}{\Delta t^2 - 2\Delta t}$, note $1 - \frac{1}{2} \cdot \Delta t^2 \theta_i < 0$ and $\frac{4}{\Delta t^2 - 2\Delta t} < \frac{6}{\Delta t^2} (\Delta t < -2)$, the stability condition requires

$$\frac{2}{\Delta t^2} < \theta_i < \frac{4}{\Delta t^2 - 2\Delta t}, \quad \Delta t < -2. \tag{34}$$

Last for (a4), $\theta_i < 0$, note $1 - \frac{1}{2} \cdot \Delta t^2 \theta_i < 0$, which are contradict apparently.
The condition (i) is obtained from Eqs. (33) and (34).

- (b) If $\Delta \geqslant 0$, exists $\Delta t \geqslant -2$. By inspecting Eq. (32), note $\Delta t \geqslant -2$, exists two situations (b1) and (b2).
- (b1) If $-2 \leqslant \Delta t < 0$, exists $\theta_i < s_1(<0)$, $\theta_i > s_2(<0)$;
- (b2) If $0 < \Delta t$, exists $\theta_i < s_2(>0)$, $\theta_i > s_1(>0)$.
- (b1) For $\theta_i < s_1$, exists $0 < -\frac{4 + \sqrt{8\Delta t + 16}}{\Delta t} < 1 - \frac{1}{2}\Delta t^2 \theta_i$. Consider Eq. (28), exists $\lambda_1 > \lambda_2 > 0$, and the stability condition requires $\lambda_1 < 1$, note $-2 \leqslant \Delta t < 0$, so $\theta_i < 0$. Thus the stability condition requires

$$\theta_i < 2 \cdot \frac{(\Delta t + 4) + \sqrt{8\Delta t + 16}}{\Delta t^3}, \quad -2 \leqslant \Delta t < 0. \tag{35}$$

For $\theta_i > s_2$. If $s_2 < \theta_i < 0$, exists $1 - \frac{1}{2}\Delta t^2 \theta_i > 0$, thus $\lambda_1 > \lambda_2 > 0$, and the stability condition satisfies $\lambda_1 < 1$. If $\frac{2}{\Delta t^2} > \theta_i > 0$, exists $1 - \frac{1}{2}\Delta t^2 \theta_i > 0$, thus $\lambda_1 > 0 > \lambda_2, |\lambda_1| > |\lambda_2|$, and the stability condition satisfies $\lambda_1 < 1$. If $\frac{2}{\Delta t^2} < \theta_i$, exists $1 - \frac{1}{2}\Delta t^2 \theta_i < 0$, thus $\lambda_1 -> 0 > \lambda_2, |\lambda_1| < |\lambda_2|$, and the stability condition satisfies $\lambda_2 > -1$. Among these situations, if $\lambda_1 < 1$, exists $\theta_i < 0$, so the stability condition requires

$$2 \cdot \frac{(\Delta t + 4) - \sqrt{8\Delta t + 16}}{\Delta t^3} < \theta_i < 0, \quad -2 \leqslant \Delta t < 0. \tag{36}$$

If $0 > \lambda_2 > -1$, exists $\theta_i < \frac{4}{\Delta t^2 - 2\Delta t}$, note $\theta_i < \frac{6}{\Delta t^2}$ and $\frac{4}{\Delta t^2 - 2\Delta t} < \frac{6}{\Delta t^2} (-2 \leqslant \Delta t < 0)$, exists $\theta_i < \frac{4}{\Delta t^2 - 2\Delta t}, -2 \leqslant \Delta t < 0$. However, $\frac{4}{\Delta t^2 - 2\Delta t} < \frac{2}{\Delta t^2}$ in $-2 \leqslant \Delta t < 0$, so it is false.

The condition (ii) is obtained from Eqs. (35) and (36).

(b2) For $\theta_i < s_2$, exists $0 < -\frac{4 - \sqrt{8\Delta t + 16}}{\Delta t} < 1 - \frac{1}{2}\Delta t^2 \theta_i$. If $0 < \theta_i < s_2$, exists $\lambda_1 > 0 > \lambda_2, |\lambda_1| > |\lambda_2|$, so the stability condition requires $\lambda_1 < 1$. So, $\theta_i > 0$ is obtained. If $\theta_i < 0$, exists $\lambda_1 > \lambda_2 > 0$, and stability condition requires $\lambda_1 < 1$. With $-\frac{2}{\Delta t^2} < \theta_i$, exists $\theta_i > 0$, which contradicts the assumption. So the stability condition requires

$$0 < \theta_i < 2 \cdot \frac{(\Delta t + 4) - \sqrt{8\Delta t + 16}}{\Delta t^3}, \quad \Delta t > 0. \tag{37}$$

For $\theta_i > s_1$, exists $-\frac{4 + \sqrt{8\Delta t + 16}}{\Delta t} > 1 - \frac{1}{2}\Delta t^2 \theta_i$. With $\lambda_1 < 1$, exists $\lambda_2 < \lambda_1 < 0$, and the stability condition requires $\lambda_2 > -1$. With $\lambda_2 > -1$, if $0 < \Delta t < 2$, exists $\frac{4}{\Delta t^2 - 2\Delta t} (< 0) < \theta_i$, else if $\Delta t > 2$, exists $\theta_i < \frac{4}{\Delta t^2 - 2\Delta t} (> 0)$. Note for $\Delta t > 2$, exists $\frac{4}{\Delta t^2 - 2\Delta t} > s_1$ anyway, so it is false.

Therefore, the stability condition requires

$$2 \cdot \frac{(\Delta t + 4) + \sqrt{8\Delta t + 16}}{\Delta t^3} < \theta_i, \quad 0 < \Delta t < 2. \tag{38}$$

$$2 \cdot \frac{(\Delta t + 4) + \sqrt{8\Delta t + 16}}{\Delta t^3} < \theta_i < \frac{4}{\Delta t^2 - 2\Delta t}, \quad \Delta t > 2. \tag{39}$$

Obviously, Eqs. (37)–(39) correspond to conditions (iii), (iv) and (v), respectively. So far all conditions are deduced. □

Theorem 4.3. *Under conditions of Lemmas 4.1, 4.2 and 4.3, for Eq. (26), $\{X_i(j), j = 1, 2, \ldots\}$ will converge to X_i^{best}.*

Proof. According to conditions from three lemmas, eigenvalues of Eq. (27) lie inside the unit cycle in complex plane. For all particles, they will converge to limit points, $\lim_{j \to \infty} X_i(j) = X_i^*, \ i = 1, 2, \ldots, N_p$. To take the limit on both sides of Eq. (26), note $\lim_{j \to \infty} \phi_i(j) = \phi_i^*, \ i = 1, 2, \ldots, N_p$, one has

$$X_i^* + \left(\frac{1}{2} \cdot \Delta t^2 \theta_i - 1\right) X_i^* + \theta_i \Delta t \cdot X_i^* = \left(\frac{1}{2}\Delta t^2 + \Delta t\right) \phi_i^* \tag{40}$$

To simplify Eq. (40) as

$$\theta_i \cdot X^* = \phi_i^*. \tag{41}$$

Substitute Eqs. (21) and (22) into Eq. (41), one has

$$G \cdot \sum_{k \in M_i} \frac{[f(X_k^*) - f(X_i^*)]^\alpha}{\|X_k^* - X_i^*\|^\beta} X_i^* = G \cdot \sum_{k \in M_i} \frac{[f(X_k^*) - f(X_i^*)]^\alpha}{\|X_k^* - X_i^*\|^\beta} X_k^*. \tag{42}$$

As assumed particles' positions are always positive. To take the log on both sides of Eq. (42)

$$\sum_{k \in M_i} \{\alpha \cdot \ln(|f(X_k^*) - f(X_i^*)|) - \beta \ln(\|X_k^* - X_i^*\|) + \ln(X_i^*)\}$$
$$= \sum_{k \in M_i} \{\alpha \cdot \ln(|f(X_k^*) - f(X_i^*)|) - \beta \ln(\|X_k^* - X_i^*\|) + \ln(X_k^*)\} \tag{43}$$

The number of elements in set M_i is cited by m_i. Eq. (43) is simplified as

$$\sum_{k \in M_i} \ln(X_i^*) = \sum_{k \in M_i} \ln(X_k^*) \Rightarrow \ln(X_i^*) = \frac{\sum_{k \in M_i} \ln(X_k^*)}{m_i}. \tag{44}$$

Table 2
Maximized results of unimodal benchmark functions.

Test functions		Global optima	SCFO				ECFO			
			Particles	γ	Iterations	Optima	Particles	γ	Iterations	
Unimodal functions	F1	0	−7.9891e−7	10	0.6	**351**	**−7.5137e−7**	7	0.6	**351**
	F2	0	−7.6076e−7	6	**0.18**	**481**	**−4.8954e−7**	8	0.54	511
	F3	0	−9.821e−7	20	0.6	401	**−9.7713e−7**	26	**0.54**	**371**
	F4	0	**−7.932e−7**	8	**0.72**	**531**	−9.7203e−7	14	**0.48**	571
	F5	0	−8.5144e−7	10	**0.54**	361	**−4.3433e−5**	28	0.6	**291**
	F6	0	−4.4384e−7	14	0.6	401	**−2.2016e−7**	10	**0.54**	**351**
	F7	0	4.655e−5	6	0.18	382	**−2.525e−5**	6	**0.06**	**111**

Table 3
Maximized results of multimodal benchmark functions.

Test functions		Global optima	SCFO				ECFO			
		Optima	Particles	γ	Iterations	Optima	Particles	γ	Iterations	
Multimodal functions	F8	837.9658	837.9657	18	0.96	371	837.9657	18	**0.06**	**291**
	F9	0	**−5.6063e−5**	12	0.66	361	−7.5095e−5	12	**0.54**	**271**
	F10	0	−8.0326e−5	8	**0.42**	**391**	**−6.8426e−5**	8	0.84	411
	F11	0	**−4.2729e−5**	14	0.66	**351**	−4.6279e−5	24	**0.12**	391
	F12	0	−3.8558e−5	18	0.66	**391**	**−1.6471e−5**	36	**0.48**	471
	F13	0	**−4.625e−5**	6	**0.54**	**311**	−6.1817e−5	18	**0.54**	371

The limit states $X_k^*, k \in M_i$ reach the optima X_i^{best} simultaneously, that is $X_1^* = X_2^* = \cdots = X_{M_i}^* = X_i^{best}$. Otherwise, according to Eqs. (24) and (25), exists $\lim_{j \to \infty} A_i(j+1) \neq 0$, which contradicts the stability case ($\lim_{j \to \infty} A_i(j+1) = 0$).

Therefore, Eq. (44) is rewritten as

$$\ln\left(X_i^*\right) = \ln\left(X_k^*\right) = \ln\left(X_i^{best}\right), \quad k \in M_i.$$

So $\{X_i(j), j = 1, 2, \ldots\}$ will converge to X_i^{best}. □

Remark 4.1 (*The Convergence of ECFO*). In retrospect SCFO preserves the best as analyzed in [28] until find the better to update it. Although it converges rapidly, the optimum is usually local. It will be renewed until check on different particles distributions throughout. However, ECFO seems to process the same routines, but the advantages of initial distributions are maximized by the adaptive mass.

For simplicity, no better one besides one which in predefined initial distribution will be found is assumed. The best is kept still in SCFO as the global optima. However, a better one may be found by adaptively expanding or shrinking gravitational range of the best in manipulation of global fitness information. To say the least, the final best in Theorem 4.3 is not the best in initial distribution as SCFO. Therefore, the initial distribution of each process in ECFO is harnessed to its full potential; meanwhile the initializations of particles distribution are reduced greatly. What's more, the searching behaviors of particles in ECFO are more complex than SCFO as in Remark 3.4, in which the global exploration will be enhanced adequately. □

5. Numerical experiments

In this section, the performance of ECFO is compared with SCFO using a suite of the former thirteen benchmark functions in [40] with two dimensions. Because of SCFO's maximum, a negative sign is applied on them to get maximum benchmark functions. The test functions include a range of different decision space topologies, which provide a general comparison.

Unlike the different results obtained in each run in scholastic algorithms, SCFO gets deterministic and accurate results under predefined parameters no matter how many times running [17,25,27,40,41]. The parameters of SCFO and ECFO are set the same. Total particles are $2 \leqslant Np \leqslant 20$ increased by 1 each time. The maximum iteration times are $N_t = 1000$ and $\Delta t = 1$. The initial distribution is a uniform distribution on "probes line", which is an axis frame based on a new origin on diagonal defined by γ [17]. Smaller γ designates less initial distribution computation. The initial γ is set to 0.06, and increasing step is set as 0.06. Another parameter is retrieve factor F_{rep} [17], initialize 0.06, and increase by 0.06.

Define $\omega_{max} = (f(X_{max}) - f(X_{min}))^\alpha$, $\chi_{min} = \min\{\|X_{max} - X_{min}\|^\beta\}$, G in SCFO is updated by the $4\chi_{min}/N_p \cdot \omega_{max}$ adaptively [28]. The default G, α, β is 2.

Thirteen benchmark functions are divided into unimodal functions ($F_1 - F_7$) and multimodal functions ($F_8 - F_{13}$). For unimodal functions, the convergence rate is more important than the final results because special methods are designed to optimize them. However, for multimodal functions, the final results are much crucial as it reflects the ability of algorithms to
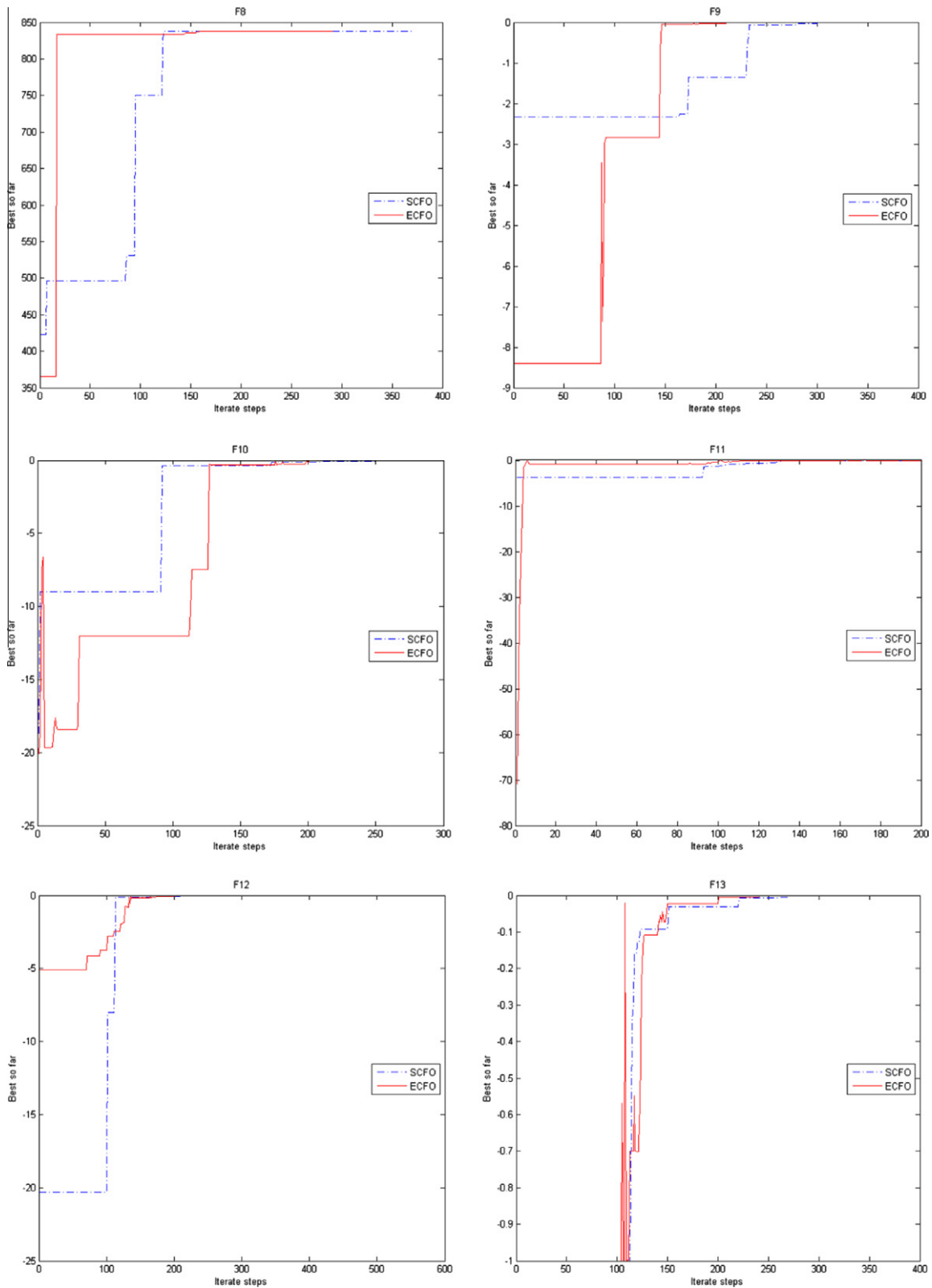
Fig. 3. Comparison between SCFO and ECFO on $F_8 - F_{13}$. The vertical axis is best function value, and the horizontal axis is the number of iterates. The solid lines indicate the results of ECFO. The dotted lines indicate the results of SCFO.

jump out from local optima or the worst. So, both the results are shown in Tables 2 and 3 and only convergence curves of $F_8 - F_{13}$ are presented in Fig. 3.

The data marked both of bold and italic in Table 2 or Table 3 indicates a better performance. The data only marked bold indicates a similar performance on this item. Table 2 shows the comparison between SCFO and ECFO on 2-D unimodal benchmark functions. For the major of unimodal functions, SCFO performs well. F_6 is a discontinuous step function, and F_7 is a noisy quadratic function, ECFO gets a higher accuracy and convergence rate than SCFO on them with a smaller γ. For others, ECFO exhibits a better performance on accuracy or low computation cost or both.

For scholastic algorithms, $F_8 - F_{13}$ is a set of difficult multimodal functions. Table 3 shows the comparison between SCFO and ECFO on them. Clearly, ECFO achieves a much better on minimum distribution factor γ with low computation cost as in Table 3. And a rapid convergence rate is obvious which is shown in Fig. 3. Hence, ECFO performs a less distribution computation than SCFO as well in accuracy.

Fig. 3 shows the typical difference of convergence speed between SCFO and ECFO. As raised in section 3, ECFO shrinks its gravitational range on F_8 and F_{11} to get a faster convergence compared with SCFO. Although ECFO expands its gravitational range on F_9 and F_{10} for global searching, it converges faster than SCFO before. ECFO oscillates at the beginning on F_{13}. After such an unstable movement, ECFO finds the optima quickly. While both SCFO and ECFO are on the stable region on F_{12}, ECFO converges to the optima steadily.

At last, the ability of "extended/enhanced" of ECFO is emphasized on expanding the gravitational range for global searching and shrinking the gravitational range for avoiding premature or the worst. By improving the mass landscape adaptively ECFO gains global fitness information to converge a much better result until reach to the actual global optima.

6. Conclusions and future works

ECFO is another novel physically-inspired heuristic deterministic optimization algorithm with more complex searching behaviors than conventional SCFO. ECFO enhances the performance of SCFO in two ways, i.e. including individual historical acceleration and defining an adaptive mass. Utilizing a new landscape of mass according to an USF based on the adaptive mean threshold, ECFO improves the exploitation of overall mass information. Merging weighted historical experience through adding historical acceleration item, the flying orbits of particles in SCFO become more complex and its global search ability is enhanced effectively. What's more, the convergence proof reveals the necessary convergence conditions of ECFO, under which it is guaranteed to converge to the optima under a predefined initial distribution. At last, to make a comparison, a suite of benchmark functions are chosen for maximizing. Experimental results show that ECFO converges faster with lower initial distribution computation cost than SCFO.

Because of the deterministic mechanism in CFO, the system stability theory is available to ECFO. Although its convergence is significant, a balance of between convergence and divergence should be emphasized. This point is similar to a balance between exploration and exploitation in traditional swarm intelligent algorithms or evolutionary algorithms. The key is how to organize the convergence condition and divergence condition to get a proper optimization performance for different - landscapes of optimization problems [28]. Besides, the gravitational metaphors are fascinating. Some other physical phenomena, e.g. dark energy, black hole and supernova, may implicate new physically-inspired heuristics. How to simplify such principles to propose or improve physically-inspired optimization algorithms is a significant and novel research work.

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